Analysis of Primes Less Than a Trillion

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Abstract— An exhaustive analysis of the highly irregularly distributed prime numbers in the range of 1 to 1 trillion is presented in this work. Since we widely use the decimal number system, the distribution trends of the primes in the blocks of all powers of 10 are explicitly explored. This is expected to throw some light on the sought hitherto unknown pattern of prime distribution.

Keywords- prime numbers; prime density; spacing; block-wise distribution

Mathematics Subject Classification 2010- 11A41, 11N05, 11N25.

I. INTRODUCTION

A prime number is an integer greater than 1 which has only two positive divisors, viz., 1 and itself. That the number of primes is infinite is known from time of Euclid and has various proofs with different interesting approaches. The needed fast determination techniques for primes have been gradually designed in earlier works [3].

II. PRIMES DENSITY

The case of all prime numbers is that their abundance gradually decreases.

The number of primes less than or equal to a given positive value x is denoted by a function $\pi(x)$.

We have computed the number of primes till one trillion, i.e., 1,000,000,000,000 (10^{12}). A dedicated algorithm was implemented in the Java Programming Language, the simple and lucid power of which is highlighted excellently in [4].

TABLE I. NUMBER OF PRIMES IN FIRST BLOCKS OF 10 POWERS.

Sr. No.	Range 1-x (1 to x)	Ten Power x	Number of Primes $\pi(x)$
1.	1-10	10^1	4
2.	1-100	10^2	25
3.	1-1,000	10^3	168
4.	1-10,000	10^4	1,229
5.	1-100,000	10^5	9,592
6.	1-1,000,000	10^6	78,498
7.	1-10,000,000	10^7	664,579
8.	1-100,000,000	10^8	5,761,455
9.	1-1,000,000,000	10^9	50,847,534
10.	1-10,000,000,000	10^{10}	455,052,511
11.	1-100,000,000,000	10^{11}	4,118,054,813
12.	1-1,000,000,000,000	10^{12}	37,607,912,018

The declining curve of percentage of the primes against the numbers for increasing powers of 10^x is approximated by power trend line of function $y = 45.89x^{-0.99}$. The vertical axis in this figure is taken on logarithmic scale with base 10.

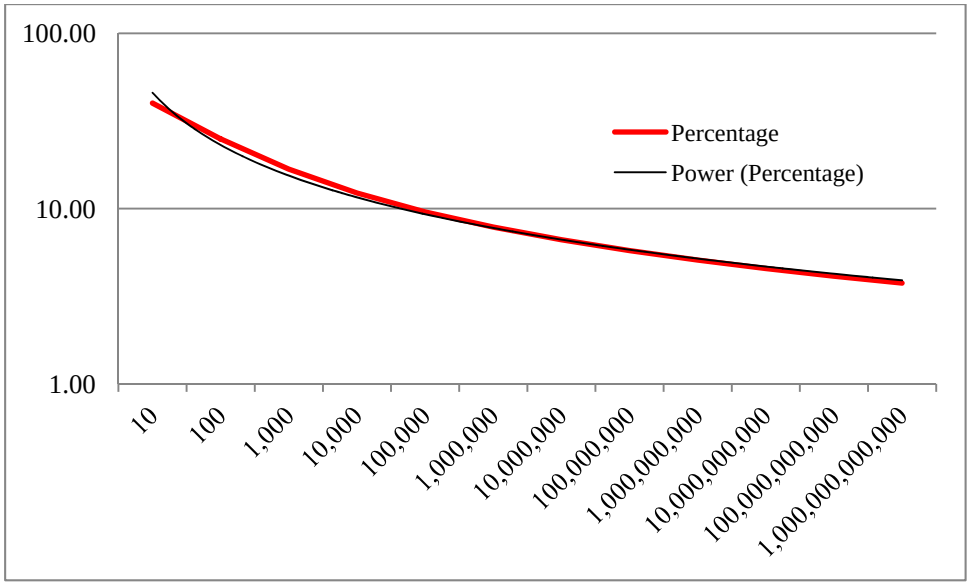

Figure 1. Percentage of primes in first blocks of 10 powers.

The reduction of number of primes in 10 blocks of 10^{11} can be estimated from the following data :

TABLE II. NUMBER OF PRIMES IN 10 BLOCKS OF 10^{11}.

Sr. No.	Range of numbers in Blocks of 10^{11}	Block Number of Size 10^{11}	Number of Primes
1.	1-1,000,000,000	1	4,118,054,813
2.	1,000,000,001-2,000,000,000	2	3,889,050,246
3.	2,000,000,001-3,000,000,000	3	3,811,334,076
4.	3,000,000,001-4,000,000,000	4	3,762,566,522
5.	4,000,000,001-5,000,000,000	5	3,727,130,485
6.	5,000,000,001-6,000,000,000	6	3,699,365,644
7.	6,000,000,001-7,000,000,000	7	3,676,572,524
8.	7,000,000,001-8,000,000,000	8	3,657,309,217
9.	8,000,000,001-9,000,000,000	9	3,640,604,059
10.	9,000,000,001-10,000,000,000	10	3,625,924,432

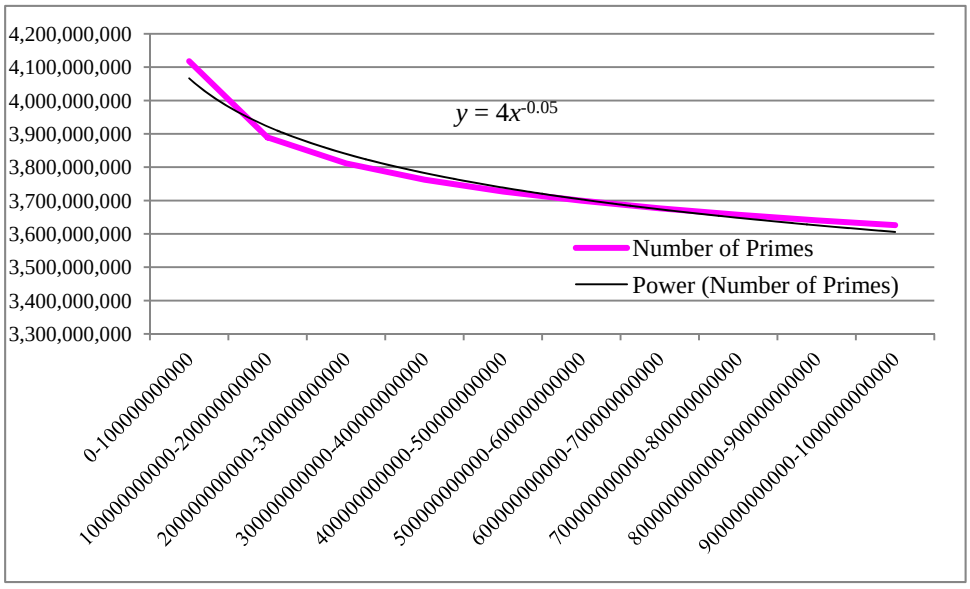

Figure 2. Percentage of primes in first blocks of 10 powers.

III. Analysis of Block-wise Distribution of Primes

Neither is there a formula to consider all primes in simple go, nor are the primes finite in number to consider them all together [1], [2]. So, to understand their random-looking distribution, we have adopted a plain approach of considering all primes up to a certain limit, viz., one trillion (10^{12}) and dividing this complete number range under consideration in blocks of powers of 10 each :

1-10, 11-20, 21-30, 31-40, · · ·

1-100, 101-200, 201-300, 301-400, · · ·

1-1000, 1001-2000, 2001-3000, 3001-4000, · · ·

⋮

A detail analysis has been performed on many fronts. Since our range is 1-10^{12}, it is clear that there are 10^{12-i} number of blocks of 10^i size for each $1 \le i \le 12$. In block-wise analysis where 10 power blocks 10^i have been used on horizontal axis, while plotting the graphs and approximating the curve by fitting function, x is the value of i instead of 10^i.

A. The First and the Last Primes in the First Blocks of 10 Powers

The inquiry of the first and the last primes in each first block of 10 powers till the range of 10^{12} under consideration is particularly interesting for the last prime, as the first prime of first power of 10 will naturally continue for all blocks ahead.

TABLE III. First and last primes in first blocks of 10 powers.

Sr. No.	Blocks of Size (of 10 Power)	First Prime in the First Block	Last Prime in the First Block
1.	10	2	7
2.	100	2	97
3.	1,000	2	997
4.	10,000	2	9,973
5.	100,000	2	99,991
6.	1,000,000	2	999,983
7.	10,000,000	2	9,999,991
8.	100,000,000	2	99,999,989
9.	1,000,000,000	2	999,999,937
10.	10,000,000,000	2	9,999,999,967
11.	100,000,000,000	2	99,999,999,977
12.	1,000,000,000,000	2	999,999,999,989

While the first prime in all the first blocks has respective fixed value, the last prime in the first blocks has natural rising; strengthening the famous property about their infinitude.

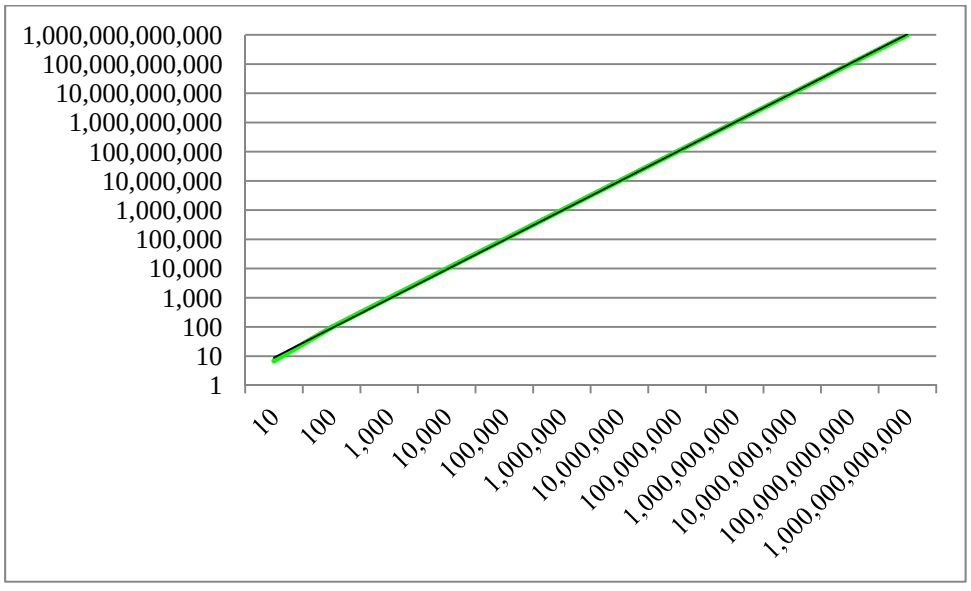

Figure 3. Last primes in first blocks of 10 powers.

The vertical axis is taken on logarithmic scale with base 10. This rising curve gets better approximated by exponential function $y = 0.879e^{2.317x}$, whose line appears in the above graph overlapping closely with the curve.

B. Minimum Number of Primes in Blocks of 10 Powers

Inspecting all blocks of each 10 power ranging from 10^1 to 10^{12} till 10^{12}, the minimum number of primes found in each 10 power block has been determined rigorously. Here block 0 means first block and consequent numbers are for higher blocks. Like for block-size 100, block 0 is 0-99, block 100 is 100 - 199 and so on.

TABLE IV. MINIMUM NUMBER OF PRIMES IN BLOCKS OF 10 POWERS.

Sr. No.	Blocks of Size (of 10 Power)	Minimum No. of Primes in Block	First Occurrence Block of Minimum No. of Primes	Last Occurrence Block of Minimum No. of Primes
1.	10	0	200	999,999,999,990
2.	100	0	1,671,800	999,999,994,200
3.	1,000	10	378,326,417,000	378,326,417,000
4.	10,000	282	946,314,330,000	946,314,330,000
5.	100,000	3,406	843,166,900,000	843,166,900,000
6.	1,000,000	35,643	986,807,000,000	986,807,000,000
7.	10,000,000	360,689	996,350,000,000	996,350,000,000
8.	100,000,000	3,616,925	997,000,000,000	997,000,000,000
9.	1,000,000,000	36,192,139	999,000,000,000	999,000,000,000
10.	10,000,000,000	361,977,421	990,000,000,000	990,000,000,000
11.	100,000,000,000	3,625,924,432	900,000,000,000	900,000,000,000
12.	1,000,000,000,000	37,607,912,018	0	0

The minimum number of primes keeps increasing with increasing inspection range.

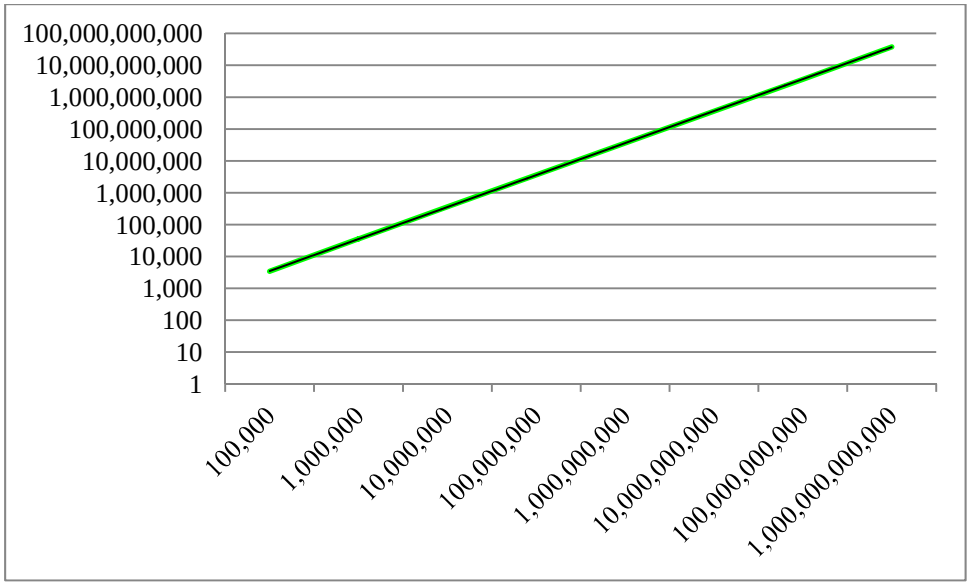

Figure 4. Minimum number of primes in blocks of 10 powers.

The first and the last 10 power blocks of the occurrences of minimum number of primes render following graphs.

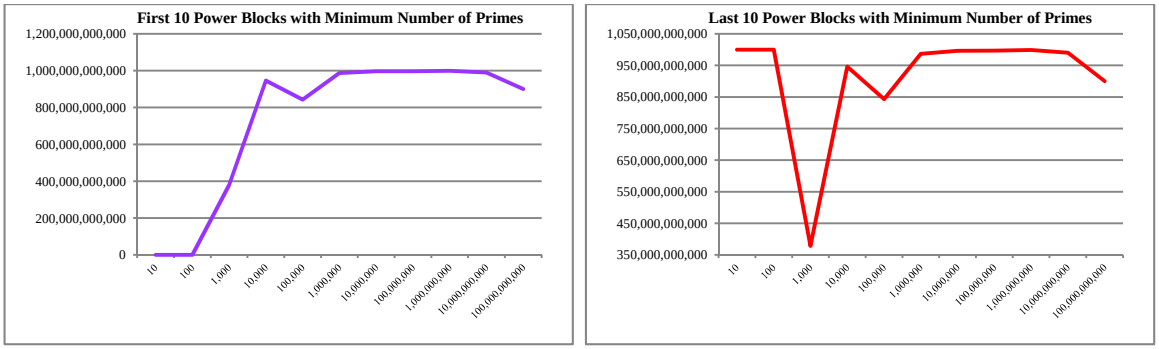

Figure 5. First and last 10 power blocks with minimum number of primes.

The number of blocks containing the minimum number of primes rapidly settles down to the unique value from size 1,000 onwards within our range of one trillion.

C. Maximum Number of Primes in Blocks of 10 Powers

For all blocks of each 10 power ranging from 10^1 to 10^{12} till 10^{12}, the maximum number of primes found in each 10 power block has also been determined rigorously.

TABLE V. MAXIMUM NUMBER OF PRIMES IN BLOCKS OF 10 POWERS.

Sr. No.	Blocks of Size (of 10 Power)	Maximum No. of Primes in Block	First Occurrence Block of Maximum No. of Primes	Last Occurrence Block of Maximum No. of Primes	Number of Blocks with Maximum No. of Primes
1.	10	4	0	999,999,843,250	8,398,278
2.	100	25	0	0	1
3.	1,000	168	0	0	1
4.	10,000	1,229	0	0	1
5.	100,000	9,592	0	0	1
6.	1,000,000	78,498	0	0	1
7.	10,000,000	664,579	0	0	1
8.	100,000,000	5,761,455	0	0	1
9.	1,000,000,000	50,847,534	0	0	1
10.	10,000,000,000	455,052,511	0	0	1
11.	100,000,000,000	4,118,054,813	0	0	1
12.	1,000,000,000,000	37,607,912,018	0	0	1

The maximum number of primes keeps increasing with increasing inspection range.

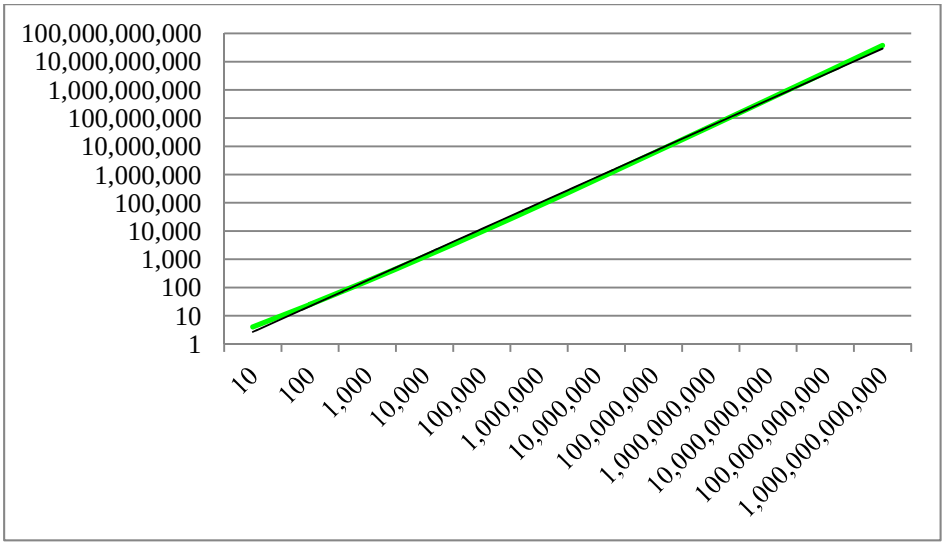

Figure 6. Maximum number of primes in blocks of 10 powers.

This is gets approximated by exponential function $y = 0.324e^{2.101x}$ for $10x$ within range of 10^{12}.

The first and the last 10 power blocks of the occurrences of maximum number of primes render following graphs.

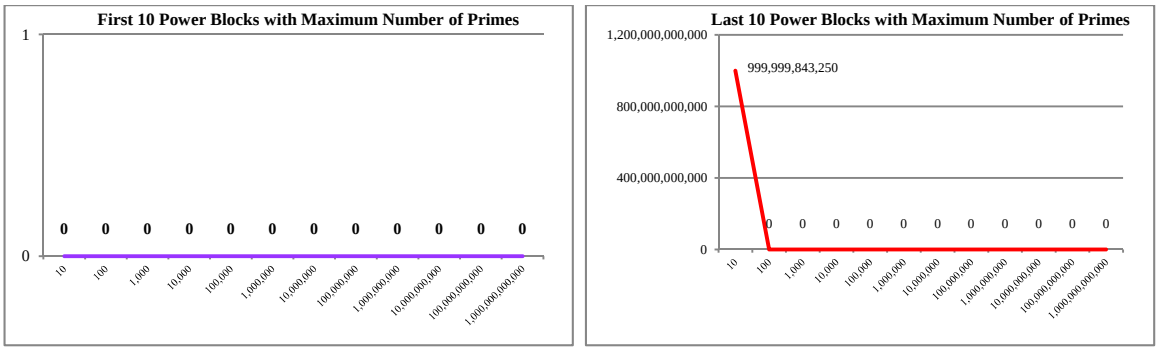

Figure 7. First and last 10 power blocks with maximum number of primes.

The number of blocks containing the maximum number of primes immediately settles down to the unique value within our range of one trillion.

IV. SPACINGS BETWEEN SUCCESSIVE PRIME PAIRS

To analyze the distribution of the primes, it is worthwhile to inspect minimum spacings between successive primes. The minimum spacing in the range of 1 to 1 trillion between successive primes is 1. This occurs between the very first prime 2 and next prime 3. In fact, in the range of 1 to 1 trillion, this minimum spacing occurs exactly once; hence the pair 2 and 3 of primes is the first as well as the last pair having minimum spacing between them in this range; and virtually till ∞.

It is equally important to delve into maximum spacings between successive primes. The maximum spacing in the range of 1 to 1 trillion between successive primes is 540. This occurs between the prime 738,832,927,927 and immediate next prime 738,832,928,467. In fact, in the range of 1 to 1 trillion, this maximum spacing occurs exactly once; hence the pair 738,832,927,927 and 738,832,928,467 of successive primes is the first as well as the last pair having maximum spacing between them in our range.

A. Minimum Spacing between Successive Primes in Blocks of 10 Powers

Exempting prime-empty blocks, the minimum spacing between successive primes in blocks of 10 powers is 1 for all blocks of all powers of 10 in all ranges, even beyond our range of a trillion, virtually till infinity! This occurs between the prime at 2 and immediate next one at 3. It occurs only once. So same is the last pair exhibiting this spacing.

B. Maximum Spacing between Successive Primes in Blocks of 10 Powers

Unlike the minimum spacing between successive primes in blocks of 10 powers, the maximum spacing in these blocks initially increases with increase in the block size and then becomes stabilized in our range of 1 to 1 trillion.

TABLE VI. MAXIMUM SPACING BETWEEN PRIMES IN BLOCKS OF 10 POWERS.

Sr. No.	Blocks of Size (of 10 Power)	Maximum Block Spacing between Successive Primes	First Prime with Respective Maximum Block Spacing	Last Prime with Respective Maximum Block Spacing	Number of Times the Maximum Block Spacing Occurs
1.	10	8	401	999,999,998,441	479,094,877
2.	100	98	18,538,001	999,998,811,401	1,404,035
3.	1,000	532	461,690,510,011	461,690,510,011	1
4.	10,000	540	738,832,927,927	738,832,927,927	1
5.	100,000	540	738,832,927,927	738,832,927,927	1
6.	1,000,000	540	738,832,927,927	738,832,927,927	1
7.	10,000,000	540	738,832,927,927	738,832,927,927	1
8.	100,000,000	540	738,832,927,927	738,832,927,927	1
9.	1,000,000,000	540	738,832,927,927	738,832,927,927	1
10.	10,000,000,000	540	738,832,927,927	738,832,927,927	1
11.	100,000,000,000	540	738,832,927,927	738,832,927,927	1
12.	1,000,000,000,000	540	738,832,927,927	738,832,927,927	1

Till our ceiling of one trillion, the following trend is seen :

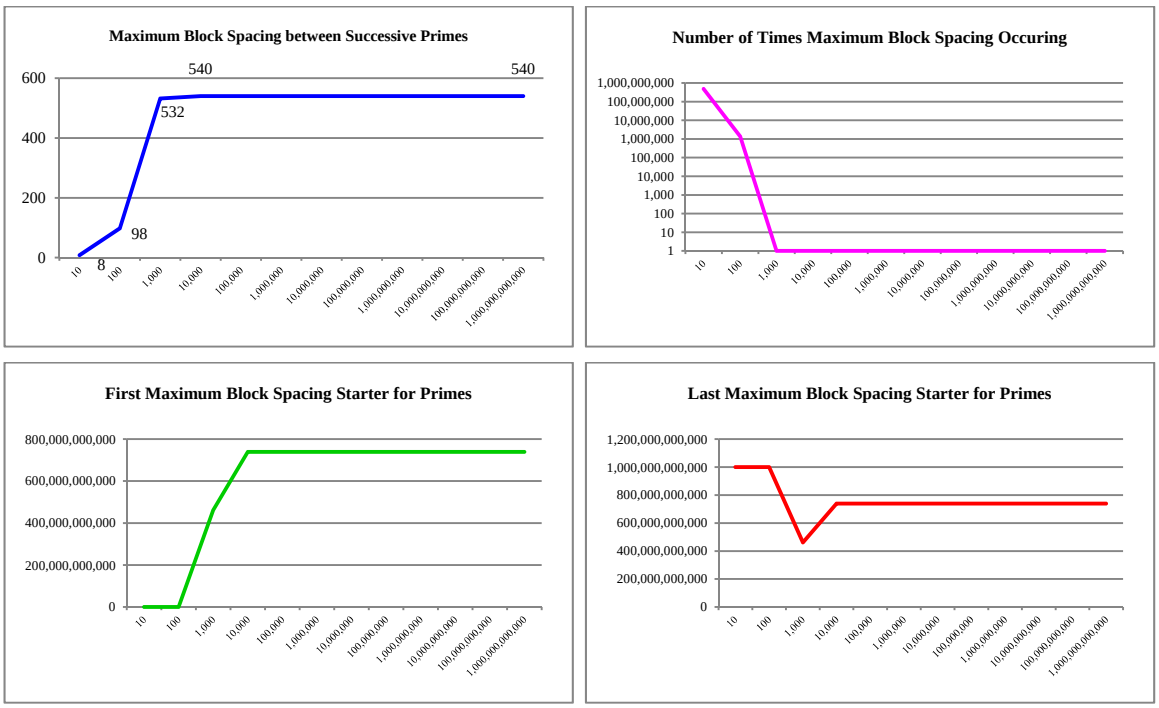

Figure 8. Maximum block spacings between successive primes.

V. END DIGITS OF PRIMES

A. *Units Place Digits in Primes*

Prime numbers can have only six different possible digits in units place. Exhaustive analysis revealed the number of primes with different digits in units place to be as follows :

TABLE VII. NUMBER OF PRIMES WITH DIFFERENT UNITS PLACE DIGITS.

Sr. No.	Digit in Units Place	Number of Primes
1.	1	9,401,960,980
2.	2	1
3.	3	9,401,979,904
4.	5	1
5.	7	9,401,997,000
6.	9	9,401,974,132

Except 2, no prime is even. Except 5, no prime can have 5 in units place. Following analysis has neglected 2 and 5 in unit places as they have exceptional nature.

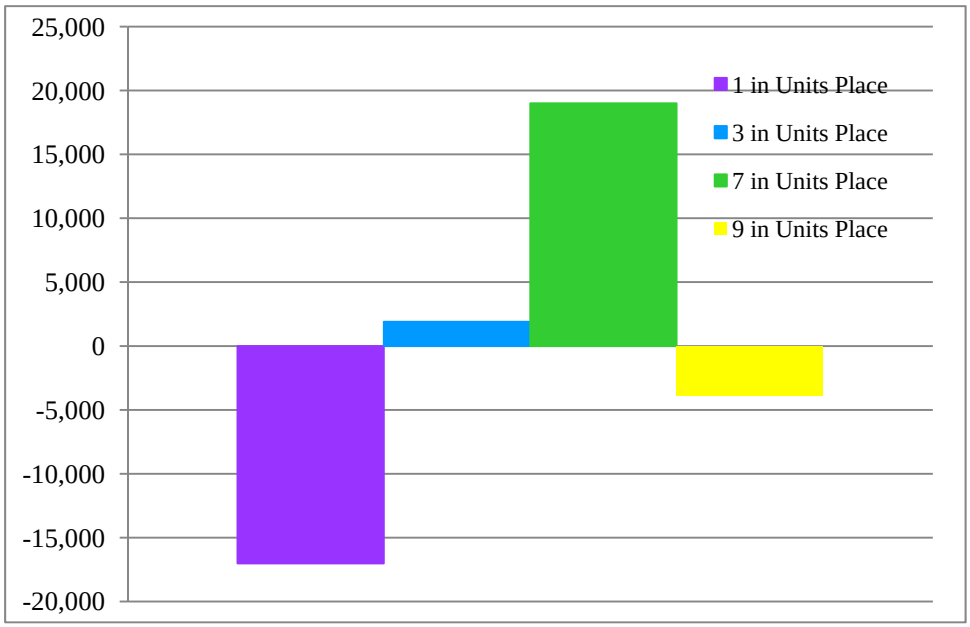

Figure 9. Deviation of number of primes with different units place digits.

B. *Tens and Units Place Digits in Primes*

There are 42 numbers possible in combination of tens and units place of any prime number. The number of primes with these is also determined in range under consideration.

TABLE VIII. NUMBER OF PRIMES WITH DIFFERENT TENS AND UNITS PLACE DIGITS.

Sr. No.	Digits in Tens & Units Place	Number of Primes
1.	01	940,201,224
2.	02	1
3.	03	940,199,042
4.	05	1
5.	07	940,201,524
6.	09	940,198,037
7.	11	940,191,631
8.	13	940,200,704
9.	17	940,189,305
10.	19	940,224,567
11.	21	940,207,451
12.	23	940,205,113
13.	27	940,207,372
14.	29	940,197,429
15.	31	940,201,296
16.	33	940,197,634
17.	37	940,198,836
18.	39	940,195,363
19.	41	940,190,006
20.	43	940,197,593
21.	47	940,197,732
22.	49	940,200,776
23.	51	940,204,880
24.	53	940,195,587
25.	57	940,192,995
26.	59	940,199,522
27.	61	940,196,110
28.	63	940,195,366
29.	67	940,203,357
30.	69	940,172,444
31.	71	940,196,489
32.	73	940,196,947
33.	77	940,196,643
34.	79	940,188,826
35.	81	940,180,003
36.	83	940,191,900
37.	87	940,199,054
38.	89	940,202,008
39.	91	940,191,890
40.	93	940,200,018
41.	97	940,210,182
42.	99	940,195,160

Neglecting special cases of 02 and 05, for which only unique primes happen to be there with these last two digits, the 40 cases show following deviation from average in the range of $1\text{-}10^{12}$.

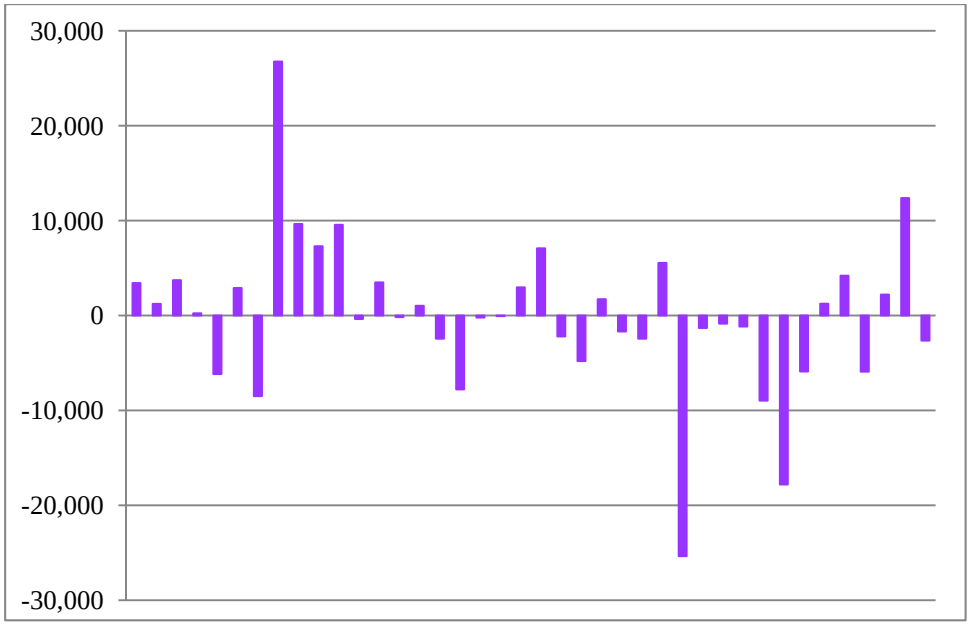

Figure 10. Deviation of number of primes with different tens and units place digits.

It is important to note that the data rigorously determined and presented here is in the range of $1\text{-}10^{12}$ only. As one progresses to higher ranges, the observed trends of the overall prime behavior may differ to more or less extent.

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