Mathematical model for Non-linear analysis of the stability of a system during unidirectional solidification with buoyancy and small segregation coefficient

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Abstract

The non-linear analysis of any physical problem is capable of predicting the qualitative and quantitative aspects of the problem. In this article, the effects of the constraints like rotation, permeability and quadratic density profile on unidirectional solidification with buoyancy and small segregation coefficient were investigated using nonlinear theory. In this study, the system consists of a fluid / fluid saturated porous layer was focused. The analysis was carried out by using a power series technique, while leading-order as well as corrections to the leading-order solutions are computed by considering higher order approximations.

Keywords: Non-Linear Analysis, unidirectional solidification, small segregation coefficient.

1.0 INTRODUCTION:

The nonlinear analysis of the problem is carried out based on the results of the linear stability analysis discussed in [2]. In this paper, a quadratic density profile which is more realistic when compared to the linear analysis discussed in [2].

2.0 MATHEMATICAL FORMULATION:

To study the nonlinear behaviour of the system by introducing [4]:

$$M=(1+\varepsilon),\quad \varepsilon<<1 \tag{9.6.1}$$

Here, ε is a parameter that measures the degree of under cooling. Thus, the whole set of equations (9.1.17) to (9.1.19) together with (9.1.21) to (9.1.27) in [2] with [1] and [3] is rescaled by using:

$$x\varepsilon^{1/2}=X,\ y\varepsilon^{1/2}=Y,\ z=Z,\ \ h=\varepsilon H,\ \ c=C,\ \varepsilon^2\bar{k}=k,\ \ t\,\varepsilon^2=T,\ \ \varepsilon^{1/2}u=U,\ \varepsilon^{1/2}v=V,\ w=W. \tag{9.6.2}$$

In framing these scales, the condition that $k/\Gamma=O(a^4)$ (9.6.3)

is taken into consideration in order to have consistency with (9.1.71) from [2]. The present analysis is restricted to two-dimensional motions only (i.e. V=0, $\frac{\partial}{\partial Y}=0$), so as to introduce the stream function ψ as:

$$U=-\frac{\partial\psi}{\partial Z},\qquad W=\frac{\partial\psi}{\partial X} \tag{9.6.4}$$

By substituting (9.6.4) with governing equations [2] and then, eliminating the pressure term in the resulting governing equations to get the equations as follows:

$$S^{-1}\left[\varepsilon^2\frac{\partial}{\partial T}+\psi_X\frac{\partial}{\partial Z}-\psi_Z\frac{\partial}{\partial X}-\frac{\partial}{\partial Z}\right](\varepsilon\psi_{XX}+\psi_{ZZ})=\left[\left(\varepsilon\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Z^2}\right)(\varepsilon\psi_{XX}+\psi_{ZZ})\right]-\varepsilon R\,C^*\frac{\partial C^2}{\partial X}\ \ldots.. \tag{9.6.5}$$

$$\varepsilon^2C_T-\psi_ZC_X+\psi_XC_Z-C_Z=\varepsilon C_{XX}+C_{ZZ} \tag{9.6.6}$$

A. *Boundary Conditions:*

At the **interface:**

$Z=\varepsilon H(X,T),\ \psi=\psi_Z=0$ …. (9.6.7); $\left\{C\left(\varepsilon^2\bar{k}-1\right)+1\right\}\left(1+\varepsilon^3 H_T\right)=C_Z-\varepsilon^2 H_XC_X$ (9.6.8)

$$\left[C-\varepsilon H(1+\varepsilon)^{-1}+\varepsilon^2\Gamma H_{XX}\left(1+\varepsilon^3H_X^2\right)^{-3/2}\right]=0 \tag{9.6.9}$$

Far-away from the interface as $Z \to \infty$,

$$|\psi| < \infty, \qquad C \to 1 \tag{9.6.10}$$

3.0 SOLUTION PROCEDURE:

The variables ψ, C, H a power series expansion in ε as follows:

$$\left.\begin{aligned} \psi &= \varepsilon\psi_1 + \varepsilon^2\psi_2 + \varepsilon^3\psi_3 + \cdots\cdots \\ C &= 1 - e^{-Z} + \varepsilon C_1 + \varepsilon^2 C_2 + \varepsilon^3 C_3 + \cdots\cdots\cdots \\ H &= H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + \varepsilon^3 H_3 + \cdots\cdots \end{aligned}\right\} \tag{9.6.11}$$

By substituting (9.6.11) with (9.6.5) to (9.6.6) and then, by equating like powers of ε to obtain a set of differential equations corresponding to different order of approximations are:

$$-S^{-1}\frac{\partial}{\partial z}\left(\psi_{1,ZZ}\right) = \frac{\partial^2}{\partial z^2}\left(\psi_{1,ZZ}\right) \text{ or } S^{-1}\psi_{1,ZZZ} + \psi_{1,ZZZZ} = 0 \tag{9.6.12}$$

$$\psi_{1,X}\left(1 - e^{-Z}\right)_Z - C_{1,Z} = C_{1,ZZ} \tag{9.6.13}$$

$$\left.\begin{aligned} &\text{With } \psi_1 = \psi_{1,Z} = 0 \text{ at } Z = 0 \\ &|\psi_1| < \infty,\ C_1 \to 1 \text{ as } Z \to \infty \\ &C_1 = 0 \text{ at } Z = 0 \end{aligned}\right\} \tag{9.6.14}$$

The solution of the above system as follows:

$$\psi_1 = 0; \quad C_1 = 0 \tag{9.6.15}$$

The higher-order approximation $\left(O\left(\varepsilon^2\right)\right)$ yields:

$$S^{-1}\psi_{2,ZZZ} + \psi_{2,ZZZZ} = 0 \tag{9.6.16}$$

$$\psi_{2,X}\, e^{-Z} - C_{2,Z} = C_{2,ZZ} \tag{9.6.17}$$

With the boundary conditions:

$$\left.\begin{aligned} &\psi_2 = \psi_{2,Z} = 0;\ C_{2,Z} + C_2 = 0 \text{ at } Z = 0 \\ &|\psi_2| < \infty,\ C_2 \to 0 \text{ as } Z \to \infty \\ &C_2 - \frac{1}{2}H_0^2 + H_0 + \Gamma H_{0,XX} = 0 \text{ at } Z = 0 \end{aligned}\right\} \tag{9.6.18}$$

The solution of the above system is as follows:

$$\psi_2 = 0 \tag{9.6.19}$$

$$C_2 = H_\alpha e^{-Z} \tag{9.6.20}$$

Where $H_\alpha = \frac{1}{2}H_0^2 - H_0 - \Gamma H_{0,XX}$ (9.6.21)

It is observed that even to this approximation, the effect of gravity and interfacial position H_0 remain undetermined. Therefore, the analysis is carried out to the third-order approximation i.e. $O\left(\varepsilon^3\right)$ and the corresponding equations are:

$$S^{-1}\psi_{3,ZZZ} + \psi_{3,ZZZZ} = 2RC^*\left(1 - e^{-Z}\right)H_{\alpha,X}e^{-Z} \tag{9.6.22}$$

$$C_{3,ZZ} + C_{3,Z} = \psi_{3,X}\, e^{-Z} - H_{\alpha,XX}e^{-Z} \tag{9.6.23}$$

Subject to the boundary conditions:

$$\left.\begin{aligned} &\psi_3 = \psi_{3,Z} = 0 \text{ at } Z = 0 \\ &C_{3,Z} + C_3 = H_{0,T} + \bar{k}\, H_0 \text{ at } Z = 0 \\ &|\psi_3| < \infty,\ C_3 \to 0 \text{ as } Z \to \infty \end{aligned}\right\} \tag{9.6.24}$$

Solutions of the above system with boundary conditions are given by:

$$\psi_3 = RC^{*}H_{\alpha,X}S\left[\frac{7}{4}+\left\{\frac{S}{2(2S-1)}-\frac{2S}{(S-1)}\right\}e^{-S^{-1}Z}+\frac{2e^{-Z}}{(S-1)}-\frac{e^{-2Z}}{4(2S-1)}\right] \quad (9.6.25)$$

$$C_3 = \Delta e^{-Z} + H_{\alpha,XX}\left[\left(1-\frac{7A_{15}}{4}\right)Ze^{-Z}+\frac{A_{15}}{(S-1)}e^{-2Z}-\frac{A_{15}}{24(2S-1)}e^{-3Z}+\frac{A_{15}S^3(3-7S)}{2(2S-1)(S^2-1)}e^{-(S+1/S)Z}\right] \quad (9.6.26)$$

Where the constant of integration Δ is determined by substituting (9.6.26) into $C_{3,Z}+C_3 = H_{0,T}+\bar{k}\,H_0$ at Z = 0 results in the evolution equation for the interval shape H_0 :

$$H_{0,T}+\bar{k}\,H_0+\left[\Gamma\,H_{0,XXXX}+\left\{(1-H_0)\,H_{0,X}\right\}_X\right]\left[1-\frac{5RC^{*}S}{6(S^{-1}+1)}\right]=0 \quad (9.6.27)$$

By setting Z=0 in (9.6.26) and then using (9.6.9) to the order of ε^3 , to obtain:

$$\Delta = H_0 + H_1(H_0-1)-\frac{H_0^3}{6}-\Gamma\,H_{1,XX}+H_{\alpha,XX}\left[A_{15}\left(\frac{1}{24(2S-1)}-\frac{1}{S-1}\right)+\frac{A_{15}S^3(7S-3)}{2(S^2 1)(2S-1)}\right] \quad (9.6.28)$$

One of the important conclusions that, whenever $RC^{*} < \frac{6}{5}(S^{-1}+1)$. (9.6.29)

Clearly, the sustained convection is absent and the transformation

$H_0 = F,\quad T=\dfrac{\Gamma\,\tau}{\left[1-\dfrac{5RC^{*}}{6(S^{-1}+1)}\right]},\quad X=\Gamma^{1/2}\xi$ .... (9.6.30) transforms (9.6.27) into the following one-parameter form of the evolution equation: $F_\tau + K_{(\Gamma)}\,F + F_{\xi\xi\xi\xi}+\left\{(1-F)\,F_\xi\right\}_\xi = 0$ (9.6.31)

Where, $K_{(\Gamma)} = \dfrac{\bar{k}\,\Gamma}{\varepsilon^2\left[1-\dfrac{5RC^{*}}{6(S^{-1}+1)}\right]}$ is the **effective segregation** coefficient. (9.6.32)

Equation (9.6.32) can be interpreted in the 3D-plane is as follows:

$$F_\tau + K_{(\Gamma)}\,F+\nabla^4F+\nabla\cdot\left[(1-F)\nabla\,F\right]=0 \quad (9.6.33)$$

Where $\nabla \equiv \frac{\partial}{\partial\xi}\hat{\mathbf{i}}+\frac{\partial}{\partial\eta}\hat{\mathbf{j}}$ (9.6.34)

and $Y=\Gamma^{1/2}\,\eta$ (9.6.35)

From (9.6.32), it is clear that in the presence of buoyancy and in the absence of sustained convection, the non-linear analysis is capable of providing an important parameter $K_{(\Gamma)}$. By a certain amount, the effective segregation coefficient which is larger than the physiochemical value of k, since k is implicitly involved in (9.6.32) and it is difficult to get a straight-forward condition in terms of R and S.

Another interesting result is that when: $RC^{*} > \frac{6}{5}(S^{-1}+1)$ (9.6.36)

the form of the evolution equation given by (9.6.33) is totally different is given below:

$$F_\tau + K_{(\Gamma)}\,F-\nabla^2F-\nabla\cdot\left[(1-F)\nabla\,F\right]=0 \quad (9.6.37)$$

This transformation is due to the interchange of stabilizing fourth-order and destabilizing second-order terms in (9.6.33). Suppose, the 2D-problem is subject to periodic boundary conditions $\xi = 0$ to $\xi = \ell$ and then by using the expansion: $C = 1-e^{-Z}+\varepsilon\,C_1+\varepsilon^2C_2+\ldots\ldots$ (9.6.38)

With $C_1 = 0$, $C_2 = H_\alpha e^{-Z}$ to obtain:

$$C = 1-e^{-Z}+\varepsilon^2\left(\frac{1}{2}F^2-F-F_{\xi\xi}\right)e^{-Z} \quad (9.6.39)$$

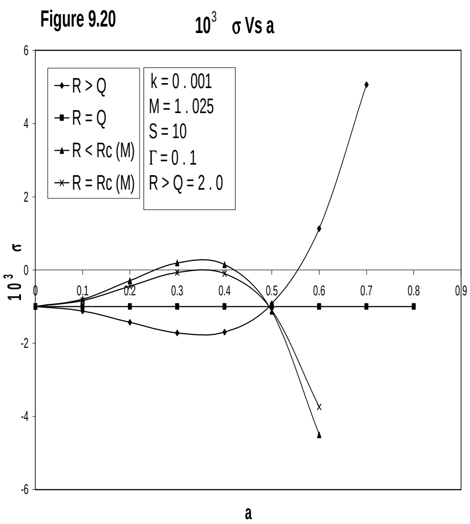
the concentration distribution provided F is known. After some simplification in (9.3.7) with $Z = \varepsilon H(X, T)$ to obtain:

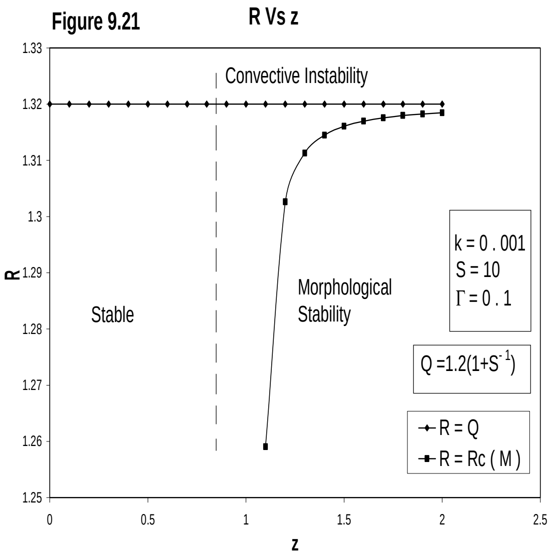
$$C = \frac{C_S}{\underset{\sim}{k}}\left[1 - \varepsilon F + \varepsilon^2\left(\tilde{\underset{\sim}{k}} + F + F_{\xi\xi}\right)\right] \tag{9.6.40}$$

This is identical with the result corresponding to linear density profile [4].

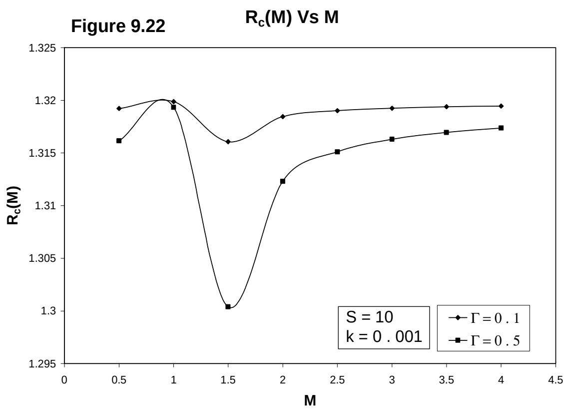
4.0 RESULTS AND DISCUSSIONS:

The results of this study are presented in the form of graphs for the wide range of parameters. The graphs corresponding to nonlinear theories are discussed in detail. The graphs revealed the following points: In figure 9.20, the plot of the growth rate expression for σ is presented for k = 0.001, S=10, M=1.025, Γ =0.1, R=1.5 and Q corresponds to the value: $Q = \frac{6}{5}\left(S^{-1} + 1\right)$. Also in figure 9.21, R>Q=2 and the behaviour can be easily studied.





In figure 9.22, the plot of R vs z is presented. For $R < \frac{6}{5}\left(1 + S^{-1}\right)$, the system attains at long times a static configuration and hence no sustained convection is possible.



Note: In all the graphs, R = RC* and $R_c = R_c C^*$.

5.0 CONCLUSION:

Based on the study, it is concluded that these graphs are of immense use in predicting the influences of the different parameters either individually or cumulatively on the morphological-convective system with small segregation coefficient and for the small wave number.

6.0 REFERENCES:

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